

Introduction to Artificial Intelligence

State Space with Costs on Actions, Heuristic Search

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<https://edux.fit.cvut.cz/courses/BIE-ZUM/>

Summary of Previous Lecture

- We learnt how to define a problem using a state space.
- We learnt some basics from the Graph Theory.
- We learnt how to search the state space without any additional information about the problem.
- Uninformed Search Algorithms:
 - ▶ Breadth-First Search (BFS)
 - ★ Expands nodes from the initial node level by level.
 - ★ Complete and optimal.
 - ★ Exponential time and space complexity.*
 - ▶ Depth-First Search (DFS)
 - ★ Expands the deepest from the opened nodes and directly travels to the deepest level of the search tree.
 - ★ Incomplete and sub-optimal.
 - ★ Exponential time complexity, but linear space complexity.*

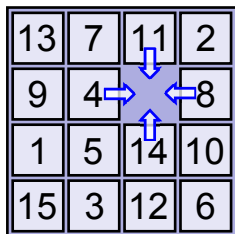
* in terms of tree search with the branching factor b and the depth d .

Costs on Actions

- Number of actions (moves) in a solution sequence is not always relevant metric for quality of a solution.
- Actions (state transitions) can differ in cost and our metric can be the total cost of a solution.
- In such case instead of sequence length we are interested in the path cost.

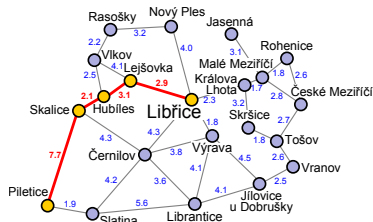
Sliding-block Puzzles

- Number of slides (actions).



Route Finding

- Length of the path (km).



State Space with Costs on Actions

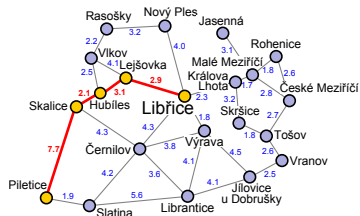
- With each action is associated a **non-negative cost**.

Definition (The State Space with Costs on Actions)

A state-space with costs on actions is a tuple $\langle S, A, \mathcal{I}, G, c \rangle$, such as (S, A) is a directed graph with a set of nodes S representing states and a set of edges A representing actions, \mathcal{I} is a set of initial states, G is a set of goal states and $c(a)$ is a cost function assigning a non negative cost to each action a .

The Cost of a Solution

- The cost of a solution is a sum of all the costs along the path i.e. the cost of the path.



$$C(p) = 7.7 + 2.1 + 3.1 + 2.9 = 15.8$$

Definition (The cost of a path)

Let $\langle S, A, \mathcal{I}, G, c \rangle$ be a state space with cost function c and let $p = (s_1, a_1, s_2, \dots, a_{n-1}, s_n)$ be an oriented path from the state s_1 to the state s_2 . The path cost is then a sum of costs of all actions along the path:

$$C(p) = \sum_{i=1}^{n-1} c(a_i).$$

Dijkstra's Algorithm

- An algorithm for finding the shortest path between two vertices in a graph.
- Actually, it finds the shortest paths
 - ▶ from a source node s_i to a goal node s_n ,
 - ▶ from a source node s_i to all other nodes in the graph.
- For each node the total path cost from the initial state to the given node is remembered.
- The node with the shortest path (lowest cost) is expanded.
- If a shorter path for already opened node is found, path cost to the node and node's predecessor is updated (i.e. the node is moved to appropriate branch in the search tree).
- Implemented using min-priority queue.

Complete: Yes

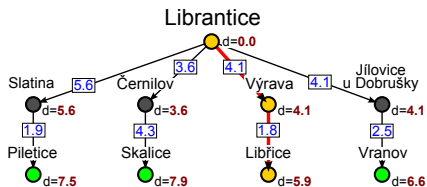
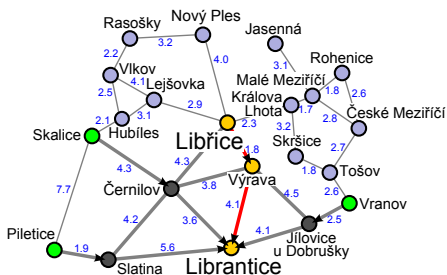
Optimal: Yes

Time: $O(b^d)$

Space: $O(b^d)$

Dijkstra's Algorithm : Example

The shortest path from Librantice to Libřice?



Algorithm 1 Dijkstra

```
1:  $Q \leftarrow \text{init\_priority\_queue}()$ 
2:  $\text{dist} \leftarrow \text{init\_table}(); \text{prev} \leftarrow \text{init\_table}()$ 
3: for all  $s \in S \setminus \{\text{init}\}$  do
4:    $\text{enqueue}(Q, s, \infty)$ 
5:    $\text{dist}[s] \leftarrow \infty; \text{prev}[s] \leftarrow \text{null}$ 
6: end for
7:  $\text{enqueue}(Q, \text{init}, 0); \text{dist}[\text{init}] \leftarrow 0$ 
8: while  $\neg \text{empty}(Q)$  do
9:    $x \leftarrow \text{dequeue}(Q)$ 
10:  if  $x \in G$  then
11:    return  $\text{reconstruct\_path}(\text{prev}, x)$ 
12:  end if
13:  for all  $y \in \text{neighbors}(x)$  do
14:     $d' \leftarrow \text{dist}[x] + c((x, y))$ 
15:    if  $d' < \text{dist}[y]$  then
16:       $\text{dist}[y] \leftarrow d'; \text{prev}[y] \leftarrow x$ 
17:       $\text{update\_key}(Q, y, d')$ 
18:    end if
19:  end for
20: end while
```

Heuristic (Informed) Search

- Uninformed search strategies rely only on a goal function.
- Informed search uses heuristic functions, that are specific to the problem and whose purpose is to guide the search towards the goal.
- Heuristics utilize some specific knowledge of the problem domain to estimate the quality or potential of partial solution.
- Heuristic algorithms are more efficient since a good heuristic function can dramatically reduce the amount of expanded nodes.
- Heuristics does not always guarantee optimal solution, however it guarantees to find a good solution in a reasonable time.

Heuristic Function

Definition (Heuristic Function)

Heuristic is any function

$$h: S \rightarrow \mathbb{R}_0^+,$$

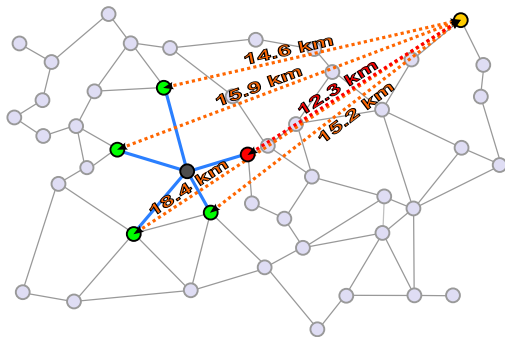
so that $h(s)$ assigns to each state $s \in S$ an estimate of the least cost path to the closest goal state $s_g \in G$ and $\forall s_g \in G : h(s_g) = 0$.

Definition (State Space with Heuristic)

A state-space with heuristic is a tuple $\langle S, A, \mathcal{I}, G, c, h \rangle$, such as (S, A) is a directed graph with a set of nodes S representing states and a set of edges A representing actions, \mathcal{I} is a set of initial states, G is a set of goal states, $c(a)$ is a cost function assigning a non negative cost to each action a and $h(s)$ is a heuristic function assigning an estimate of the least cost path to the closest goal state from each state s .

Example: Route Finding

- In the route finding problem we can use the information about geographical location between the cities.
- As a heuristic we can use the air distance to the goal city, which is the shortest possible distance.



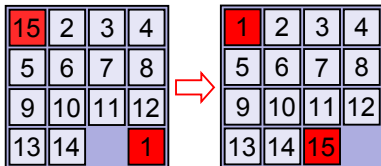
Example: Sliding Puzzles

- In the sliding puzzles problem we can use the information about the goal configuration.
- A heuristic can be manhattan distance for each tile to final position.

Manhattan distance heuristic

$$h(s) = \sum_{i=1}^N (|x_i(s) - x_i^*| + |y_i(s) - y_i^*|),$$

where $x_i(s)$ and $y_i(s)$ are the x and y coordinates of tile i in state s , and if x_i^* and y_i^* are the x and y coordinates of tile i in the goal state.



$$h(s) = 6 + 5 = 11$$

Greedy Search

- Algorithm always expands the node with the best (lowest) heuristic value

$$s^* = \arg \min_{s \in OPEN} h(s)$$

- Greedy search chooses the node that appears to be the closest to the goal since it assumes that this leads quickly to a solution.
- Success of greedy search depends on quality of heuristic function.

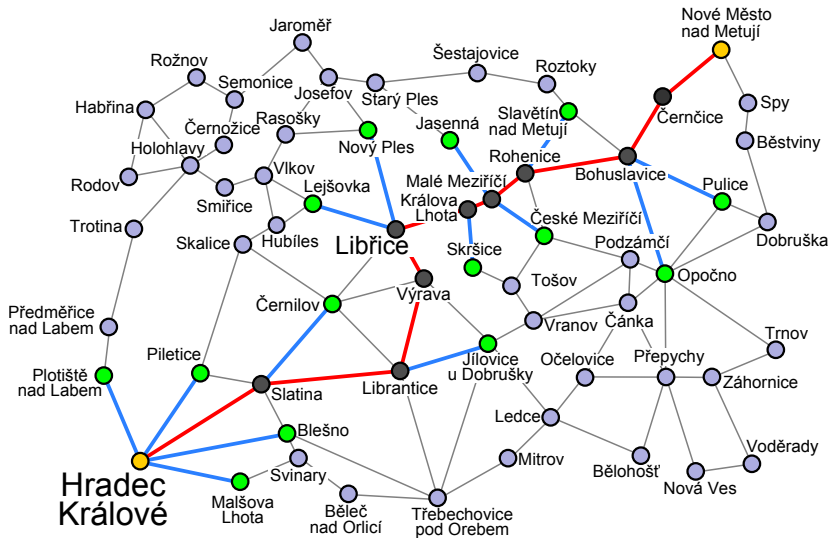
Complete: No

Optimal: No

Time: $O(b^m)$

Space: $O(b^m)$

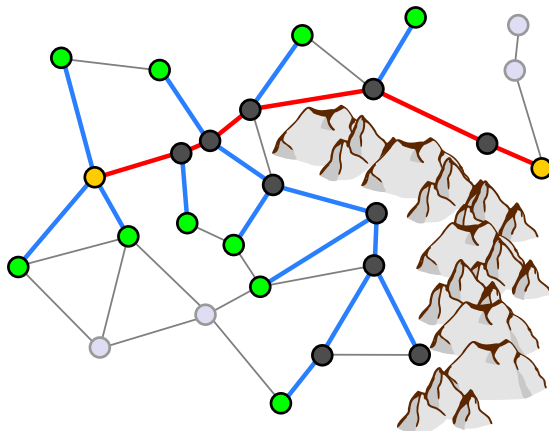
Greedy Search: Example



Algorithm 2 Greedy search

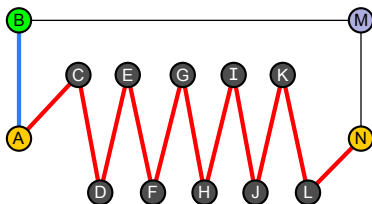
```
1:  $open \leftarrow \text{init\_priority\_queue}()$ ;  $closed \leftarrow \{\}$ 
2:  $prev \leftarrow \text{init\_table}()$ 
3: for all  $s \in \mathcal{I}$  do
4:    $\text{enqueue}(open, s, h(s))$ 
5: end for
6: while  $\neg \text{empty}(open)$  do
7:    $x \leftarrow \text{dequeue}(open)$ 
8:   if  $x \in G$  then
9:     return  $\text{reconstruct\_path}(prev, x)$ 
10:  end if
11:  for all  $y \in \text{neighbors}(x)$  do
12:    if  $y \notin (open \cup closed)$  then
13:       $\text{enqueue}(open, y, h(y))$ 
14:       $prev[y] \leftarrow x$ 
15:    end if
16:  end for
17:   $closed \leftarrow closed \cup \{x\}$ 
18: end while
```

Greedy Search: Air Distance Failure



Greedy Search: Failure

The following example is showing how simply can greedy search fail in some conditions:



Greedy search will step by step expand the states C, D, E, \dots, N ; state B will not be ever expanded and therefore the shortest path A, B, M, N will never be found!

A* (A-star Search)

- Combination of greedy search and Dijkstra.
- Takes into account both the heuristic and the path cost.

Complete: Yes

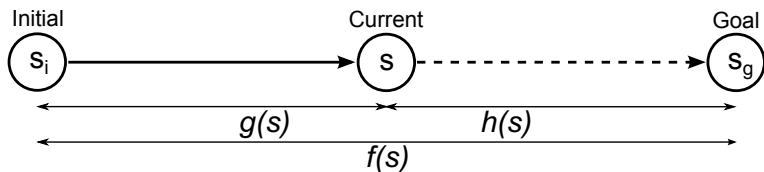
Optimal: Yes*

Time: $O(\min(b^{d+1}, b|S|))$

Space: $O(\min(b^{d+1}, b|S|))$

* if the heuristic h is optimistic.

Functions Notation



- $g(s)$ is a cost function that estimates least cost path from initial state to state s .
- $h(s)$ is a heuristic function that estimates the least cost path from state s to the goal state.
- $f(s)$ is an evaluation function that estimates least cost solution through state s .

A*

Dijkstra's expands the node with the lowest cost g :

$$s^* = \arg \min_{s \in OPEN} g(s)$$

Greedy Search expands the node with lowest heuristic value:

$$s^* = \arg \min_{s \in OPEN} h(s)$$

A* combines both approaches:

$$s^* = \arg \min_{s \in OPEN} (g(s) + h(s)) = \arg \min_{s \in OPEN} f(s)$$

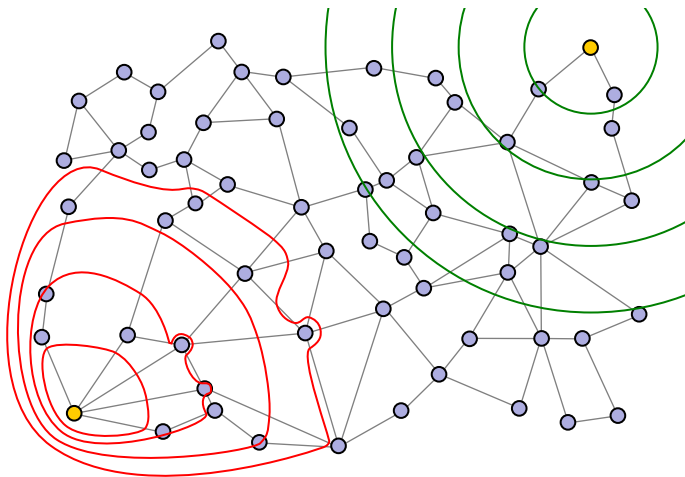
Algorithm 3 A*

```

1: open  $\leftarrow$  init_priority_queue()
2: dist  $\leftarrow$  init_table(); prev  $\leftarrow$  init_table()
3: for all  $s \in \mathcal{I}$  do
4:   enqueue(open,  $s$ ,  $h(s)$ )
5:   dist[ $s$ ]  $\leftarrow$  0; prev[ $s$ ]  $\leftarrow$  null
6: end for
7: while  $\neg$ empty(open) do
8:    $x \leftarrow$  dequeue(open)
9:   if  $x \in G$  then
10:    return reconstruct_path(prev,  $x$ )
11:   end if
12:   for all  $y \in \text{neighbors}(x) \setminus \text{closed}$  do
13:     $d' \leftarrow \text{dist}[x] + c((x, y))$ 
14:    if  $y \notin \text{open} \vee \text{dist}[y] > d'$  then
15:      dist[ $y$ ]  $\leftarrow$   $d'$ ; prev[ $y$ ]  $\leftarrow$   $x$ 
16:      if  $y \notin \text{open}$  then
17:        enqueue(open,  $y$ ,  $d' + h(y)$ )
18:      else
19:        update_key(open,  $y$ ,  $d' + h(y)$ )
20:      end if
21:    end if
22:   end for
23:   closed  $\leftarrow$  closed  $\cup$  { $x$ }
24: end while

```

A*: Real vs Heuristic Contours



Optimistic Heuristics

Definition (Optimistic heuristic)

Heuristic function h is optimistic (or admissible) if it never overestimates, i.e., its value is never greater than the true cost needed to reach the goal

$$\forall s \in S : h(s) \leq h^*(s),$$

where $h^*(s)$ is the true path cost to the goal state from state s .

- If the heuristics is not optimistic, the search algorithm might bypass the optimal path because it seems more expensive than it really is.

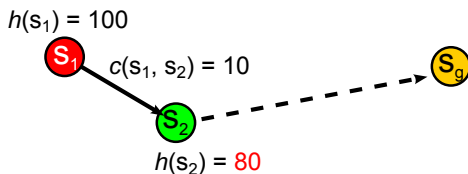
Consistent Heuristics

Definition (Consistent heuristic)

Heuristic h is consistent (or monotone) if for every state s_1 and every successor s_2 of s_1 generated by any action $a = (s_1, s_2)$, the estimated cost of reaching the goal state from s_1 is less or equal than the estimated cost of reaching the goal state from the state s_2 plus cost of the transition action a .

$$\forall (s_1, s_2) \in A : h(s_1) \leq h(s_2) + c(a)$$

Figure: Example of not consistent heuristic function (heuristic estimate decreases by 20 with move from s_1 to s_2 of cost 10)



Dominant Heuristics

Definition (Dominant heuristic)

Let A_1^* and A_2^* be optimal algorithms with optimistic heuristic functions h_1 and h_2 .
Algorithm A_1^* dominates algorithm A_2^* if

$$\forall s \in S : h_1(s) \geq h_2(s)$$

. We also say that A_1^* is more informed algorithm than A_2^*

- A more informed algorithm will generally search through a smaller state space than a less informed algorithm.

A^* Heuristics

Let h be a heuristic function

- if h is monotone
 - ▶ then h is also optimistic,
 - ▶ then A^* finds an optimal solution.
- if h is optimistic (but not monotone)
 - ▶ then A^* finds an optimal solution if it checks the path cost for already closed nodes.
- if h is not optimistic nor monotone
 - ▶ then A^* does not guarantee to find an optimal solution.

How To Build Heuristics

- Problem relaxation
 - ▶ the cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem.
- Combining several optimistic heuristics
 - ▶ If h_1, h_2, \dots, h_n are optimistic, we can combine them into a single dominant heuristic that will also be optimistic:

$$h(s) = \max(h_1(s), h_2(s), \dots, h_n(s))$$

- Using sub-problem costs
 - ▶ the cost of the optimal solution of a sub-problem is a lower bound on the cost of the complete problem,
 - ▶ database of sub-problems with known cost.
- Learning heuristics
 - ▶ using machine learning techniques to derive $h(s)$ from state features, e.g. linear combination of state features x_1, \dots, x_n

$$h(s) = \omega_1 x_1(s) + \omega_2 x_2(s) + \dots \omega_n x_n(s)$$

Good Heuristic

A good heuristic is:

- **optimistic,**
- **monotone,**
- **well informed,**
- **simple to compute.**